

## Oscillations and dynamics in a two-dimensional prey-predator system

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Using Monte Carlo simulations we study two-dimensional prey-predator systems. Measuring the variance of densities of prey and predators on the triangular lattice and on the lattice with eight neighbors, we conclude that temporal oscillations of these densities vanish in the thermodynamic limit. This result suggests that such oscillations do not exist in two-dimensional models, at least when driven by local dynamics. Depending on the control parameter, the model could be either in an active or in an absorbing phase, which are separated by the critical point. The critical behavior of this model is studied using the dynamical Monte Carlo method. This model has two dynamically nonsymmetric absorbing states. In principle both absorbing states can be used for the analysis of the critical point. However, dynamical simulations which start from the unstable absorbing state suffer from metastablelike effects, which sometimes renders the method inefficient.

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### I. INTRODUCTION

Nonequilibrium, many-body systems were usually regarded as one of the classical domains of physics. However, recently it is becoming clear that such systems are of interest also in other disciplines such as biology [1], economy [2], or sociology [3]. Consequently, notions of cooperative effects, formation of order or phase transition, which were typically restricted to physics, proliferate other scientific disciplines.

An example of a multidisciplinary problem, which has application both in biological population dynamics and in hetero- or homogeneous catalysis, is the existence of temporarily periodic solutions in the so-called prey-predator systems. A closely related problem is synchronization, which appears in various biological contexts such as heartbeats, wake-sleep cycles and many others [4]. The earliest mathematical model of the temporal oscillations in interacting populations was proposed by Lotka and Volterra, who described populations of preys and predators in terms of nonlinear differential equations [5]. However, description of interacting populations in terms of ordinary differential equations, by necessity, introduces considerable simplifications. In particular it neglects fluctuations and spatial inhomogeneities which are certainly present in such systems. One possibility to take such effects into account is to describe such systems using lattice models. The first approach of this kind was made by Chaté and Manneville [6]. Subsequently, more complicated versions of their model were studied [7]. Although in some cases these studies confirmed existence of periodic in time solutions, these models were driven by synchronous dynamics. More realistic models with asynchronous dynamics were also examined [8–10]. As we already mentioned, closely related models are studied in the context of catalysis [11].

On general grounds one expects that fluctuations in low-dimensional systems are strong enough to destroy such temporal oscillations in the thermodynamic limit. An important question is: what is the critical dimension  $d_c$  above which such oscillations will survive in the thermodynamic limit.

Certain arguments were given by Grinstein *et al.* that  $d_c = 2$  [12]. Numerical results for both synchronous [6,13] and asynchronous models [14] seem to confirm that temporal oscillations exist but only for  $d > 2$ . Recently, however, numerical simulations by Rosenfeld *et al.* suggest that in a model with smart preys and predators oscillatory behavior might exist even in  $d = 2$  systems [15]. They argued that the oscillatory phase is induced by a certain dynamical percolation transition. If this percolative mechanism would be of more general validity, then the oscillatory phase should exist also in other two-dimensional models. One would then argue that the fact that this phase was not observed in some earlier simulations [14] was due to for example too small a range of interactions of preys and predators.

In the present paper we examine two-dimensional versions of a prey-predator model [14] for the increased range of interaction. However, our simulations show that even with the increased range of interaction, an amplitude of oscillations vanishes in the thermodynamic limit. Let us notice that in three-dimensional version of the model, oscillations exist even for the nearest-neighbor interactions [14]. Thus, our result strongly indicates that, at least for our model, there are no oscillations in the case  $d = 2$ .

Usually lattice prey-predator models have an absorbing state. Consequently, they might undergo a phase transition in the steady state between active and absorbing phases, which is expected to belong to the directed percolation universality class [16]. Our model is characterized by two absorbing states. However, these states are asymmetric and only one of them is typically reached by the model's dynamics. The second absorbing state can be considered as dynamically unstable. Because of this asymmetry the model behaves as if it has only one absorbing state and thus should exhibit directed percolation criticality, which was confirmed for the one-dimensional case [17]. Critical behavior of models with absorbing states can be studied using the so-called dynamical Monte Carlo method [19]. In this method we prepare the system in an absorbing state and locally activate it. Statistics of such runs provide a very efficient method to study the

critical behavior of such models. This method was already applied to a number of models with a single absorbing state or with double but symmetric ones [20]. We are, however, not aware of studies for models with asymmetric absorbing states. We show that when dynamical simulations start from an unstable absorbing state they suffer from metastable-like effects, which in the two-dimensional case renders the method inefficient. In the one-dimensional case this effect is much weaker and both absorbing states can be used to extract information about the critical point.

In Sec. II we define the model, briefly mention the results of the simple mean-field approximations, and study the fluctuations of densities of populations. In Sec. III we present the results of the dynamical Monte Carlo method. Section IV contains our conclusions.

## II. MODEL AND ITS STEADY-STATE SIMULATIONS

In our model on each site of a  $d$ -dimensional Cartesian lattice of linear size  $N$  we have a four-state variable  $\epsilon_i=0,1,2,3$ , which corresponds to the site being empty ( $\epsilon=0$ ), occupied by a prey ( $\epsilon=1$ ), occupied by a predator ( $\epsilon=2$ ) or occupied by a prey and a predator ( $\epsilon=3$ ). Dynamics of this model is specified as follows [14]:

- (i) Choose a site at random (say  $i$ th).
- (ii) With the probability  $r$  ( $0 < r < 1$ ) update a prey at the chosen site, provided that there is one (i.e.,  $\epsilon=1$  or 3); otherwise do nothing. Provided that at least one neighbor of the chosen site is not occupied by a prey (i.e.,  $\epsilon=0$  or 2), the prey (which is to be updated) produces one offspring and places it on the empty neighboring site (if there are more empty sites, one of them is chosen randomly). Otherwise (i.e., when there is a prey on each neighboring site) the prey does not breed (due to overcrowding).
- (iii) With the probability  $1-r$  update a predator at the chosen site, provided that there is one (i.e.,  $\epsilon=2$  or 3). Provided that the chosen site is occupied by a predator but is not occupied by a prey ( $\epsilon=2$ ), the predator dies (of hunger). If there is a prey on that site (i.e.,  $\epsilon=3$ ), the predator survives and consumes the prey from the site it occupies. If there is at least one neighboring site which is not occupied by a predator, the predator produces one offspring and places it on the empty site (chosen randomly when there are more such sites).

To complete the description of this model we have to specify what are the neighboring sites, i.e., sites where offsprings can be placed. In previous studies of this model these were just nearest neighbors [14,17]. In the present paper of our main concern are  $d=2$  models with further (but finite) range.

Steady-state description of our model is given in terms of densities of preys  $x$  and predators  $y$  defined as

$$x = \frac{1}{N^d} \sum_i (\delta_{\epsilon_i,1} + \delta_{\epsilon_i,3}), \quad y = \frac{1}{N^d} \sum_i (\delta_{\epsilon_i,2} + \delta_{\epsilon_i,3}), \quad (1)$$

where summation is over all  $N^d$  sites  $i$  and  $\delta$  is Kronecker's  $\delta$  function. From the above rules it follows that the model

has two absorbing states. The first one (PREY) is filled with preys ( $x=1, y=0$ ) and the second one (EMPTY) is empty ( $x=y=0$ ). For large enough  $r$ , both populations coexist and the model is in the active phase ( $x>0, y>0$ ). When the update rate of preys  $r$  decreases, their number become small to support predators. For sufficiently small  $r$  predators die out and the system quickly reaches the absorbing state PREY. In the simplest case, when the neighboring sites are nearest neighbors, the phase transition between active and absorbing phase was observed at positive  $r$  for  $d=1,2,3$  [17]. For  $d=1$  (linear chain) Monte Carlo simulations show that, as expected, the phase transition belongs to the directed percolation universality class.

To get additional insight into the behavior of the model we can use a mean-field approximation. In its simplest version one describes the model solely in terms of densities  $x$  and  $y$  [which are strictly speaking time average quantities defined in Eq. (1)]. From the dynamics of the model and typical mean-field assumptions one can easily arrive at the following equations [14,17]:

$$\frac{dx}{dt} = rx(1-x^w) - (1-r)xy, \quad (2)$$

$$\frac{dy}{dt} = (1-r)xy(1-y^w) - (1-r)y(1-x), \quad (3)$$

where  $w$  is the number of neighboring sites [see the dynamical rule (ii) and (iii)]. Predictions of such an approximation is, however, substantially different from the behavior of the model as observed in Monte Carlo simulations [14]. In particular, approximations (2) and (3) fail to predict a phase transition between active and absorbing phases at positive  $r$  and for any dimension. Moreover, there is no indication of the oscillatory phase as observed in the  $d=3$  case.

An improved version of the mean-field approximation can be obtained introducing a third variable  $z$ , which denotes a density of sites occupied by a prey and a predator ( $\epsilon=3$ ). Then, the mean-field equations are written as

$$\frac{dx}{dt} = rx(1-x^w) - (1-r)z, \quad (4)$$

$$\frac{dy}{dt} = (1-r)z(1-y^w) - (1-r)(y-z), \quad (5)$$

$$\begin{aligned} \frac{dz}{dt} = & \frac{rx(1-x^w)(y-z)}{1-x} \\ & - \frac{(1-r)z(1+z-x-y)(1-y^w)}{1-y} \\ & - (1-r)zy^w. \end{aligned} \quad (6)$$

In the approximations (2) and (3) the density of sites that are occupied by a prey and a predator is simply given by the product  $xy$ . In the approximation (4)–(6) this is an independent variable, whose time evolution follows from the dynamical rules of the model. Numerical solution of Eqs. (4)–

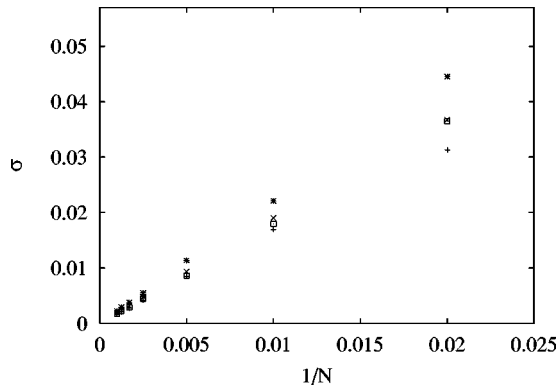


FIG. 1. The variance of the density of preys as a function of  $1/N$  for the model on the triangular lattice. Simulations were made for  $r=0.4$  ( $\square$ ),  $0.3$  ( $*$ ),  $0.2$  ( $\times$ ), and  $0.15$  ( $+$ ).

(6) shows that this approximation is indeed more accurate [18]. In particular, it predicts that for  $w \geq 4$  (which could be interpreted as  $d \geq 2$ ) there is a range of  $r$  in the active phase where nonvanishing oscillations exist. For  $w=4$  (i.e., square lattice) this is still at odds with Monte Carlo simulations [14]. Let us emphasize that both Monte Carlo simulations and mean-field calculations are essentially independent on the initial configuration (provided, of course, we do not start from an absorbing state).

To examine the oscillatory behavior, one can measure the standard deviation of the densities defined as  $\sigma_x = \sqrt{\langle (x - \overline{x})^2 \rangle}$  and analogously for the densities of predators  $y$ . The symbol  $\langle \dots \rangle$ , denotes time average in the steady state. It was found that for  $N \rightarrow \infty$   $\sigma = \sigma_x$  always decreases to zero for  $d=1,2$  [14]. However, for  $d=3$  there is a certain range of  $r$  in the active phase and such that  $\sigma$  remains finite for  $N \rightarrow \infty$ . Such a behavior indicates that finite-amplitude oscillations survive in the thermodynamic limit.

To check whether an increased range of interactions might stabilize oscillations in two-dimensional systems, we simulated our model on the triangular lattice and on the lattice with 8 neighbors (square lattice with nearest- and next-nearest neighbors). On such lattices the parent site that is supposed to breed looks for an empty site among 6 or 8 neighboring sites, respectively. On such lattices we also observed a phase transition into absorbing phase around  $r \sim 0.06-0.08$ . Similarly to other lattices [14], for  $r$  larger but close to the transition point fluctuations of densities exhibit oscillatory-like behavior. However, analysis of the variance of this fluctuations shows (see Fig. 1 and Fig. 2) that in the thermodynamic limit amplitude of oscillations vanishes. Simulations were made for  $N \leq 1000$  and simulation time was long enough to ensure that error bars are smaller than the plotted symbols.

Let us notice that for the lattice with 8 neighbors the coordination number is greater than that in the three-dimensional (cubic lattice) case. While in the former case oscillations die out in the thermodynamic limit, they survive in the latter one [14]. It indicates that in two-dimensional systems oscillations are unlikely. However, such a conclusion, if true, is applicable only to models equipped with local dynamics.

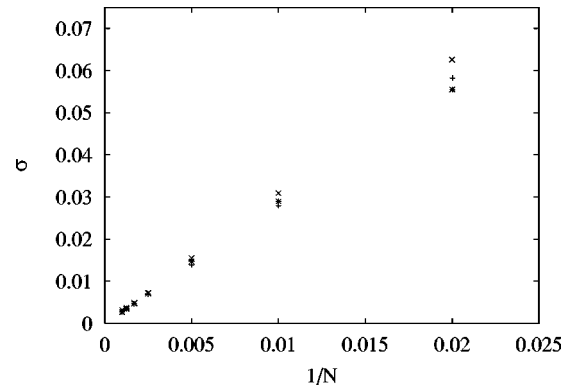


FIG. 2. The variance of the density of preys as a function of  $1/N$  for the model on the lattice with nearest- and next-nearest neighbors. Simulations were made for  $r=0.35$  ( $*$ ),  $0.3$  ( $\times$ ), and  $0.25$  ( $+$ ).

One factor which is omitted in our approach is diffusion. Our organisms diffuse only effectively through the breeding process. In related two-dimensional models, which were studied in the context of heterogeneous catalysis, diffusion was taken into account [21]. It seems, however, that as long as the diffusion constant is finite, the amplitude of oscillations vanishes in the thermodynamic limit.

### III. DYNAMICAL MONTE CARLO

The essence of this method is to prepare the system in an absorbing state and to activate it locally [19]. Quantities of the main interest are the probability  $P(t)$  that the activity did not die out until time  $t$  (the unit of time is defined as a single, on average, update of each site) and the average number of active sites  $N(t)$  (averaged over all runs). One expects that at criticality these quantities have the power-law behavior:  $P(t) \sim t^{-\delta}$ ,  $N(t) \sim t^\eta$ , where  $\delta$ , and  $\eta$  are critical exponents characteristic to a given universality class. Off the critical point  $P(t)$  and  $N(t)$  deviate from the power-law behavior.

We applied the dynamical Monte Carlo method to the two-dimensional model with nearest neighbor interactions

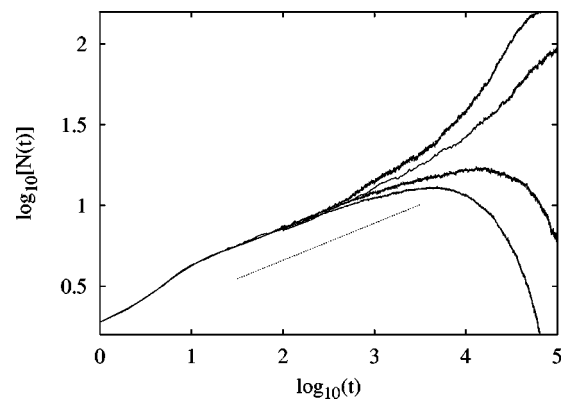


FIG. 3. The average number of predators  $N(t)$  as a function of time  $t$  for the model on the square lattice with the absorbing state PREY. Simulations were made for (from top)  $r=0.1099$ ,  $0.1098$ ,  $0.1097$ , and  $0.1096$ . The dotted line has a slope  $0.23$  which corresponds to the exponent  $\eta$  in (2+1) DP [20].

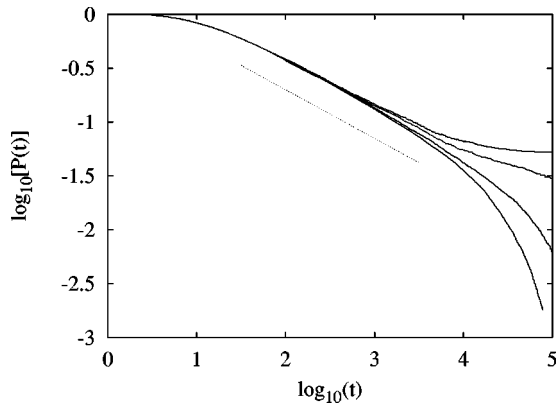


FIG. 4. The survival probability of predators  $P(t)$  as a function of time  $t$  for the model on the square lattice with the absorbing state PREY. Simulations were made for (from top)  $r=0.1099, 0.1098, 0.1097,$  and  $0.1096$ . The dotted line has a slope  $0.451$  which corresponds to the exponent  $\delta$  in  $(2+1)$  DP [20].

(square lattice). In this case the steady-state calculations show that the model undergoes a transition at  $r=r_c \sim 0.11$ . Our results in the case of the absorbing state PREY are shown in Fig. 3 and Fig. 4. As an initial configuration we put all sites in a prey state ( $\epsilon=1$ ), except one, randomly selected, site which is in a prey-predator state ( $\epsilon=3$ ). Actually, in our model there are two quantities which scale approximately in the same way, namely, the number of predators and the number of active sites (a site is active if it contains a predator, a prey and predator or a prey which is surrounded by at least one site without a prey). To calculate  $P(t)$  and  $N(t)$  we considered only the number of predators (when this number becomes zero we stop a given trial) but qualitatively the same results are obtained when the number of active sites is considered. Using the dynamical Monte Carlo method it is important to ensure that the spreading activity does not reach the boundary of the system. We checked that the value which we used ( $N=2000$ ) was sufficiently large. From the behavior of  $P(t)$  and  $N(t)$  we con-

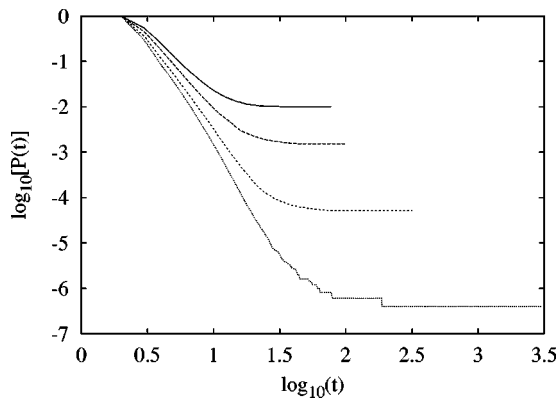


FIG. 5. The survival probability of predators  $P(t)$  as a function of time  $t$  for the model on the square lattice with the absorbing state EMPTY. Simulations were made for (from top)  $r=0.25, 0.2, 0.15,$  and  $0.12$ .

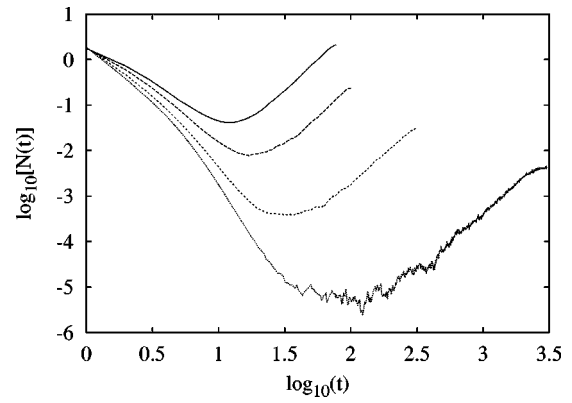


FIG. 6. The average number of predators  $N(t)$  as a function of time  $t$  for the model on the square lattice with the absorbing state EMPTY. Simulations were made for (from top)  $r=0.25, 0.2, 0.15,$  and  $0.12$ .

clude that the phase transition is located at  $r=r_c = 0.10975(5)$ , which is far more accurate than the steady-state estimation. Moreover, one can see that our data upon approaching the critical point have a power-law behavior with exponents consistent with  $(2+1)$  directed percolation [20].

Now, let us apply the dynamical Monte Carlo in the case of the EMPTY absorbing state. As an initial configuration we put all sites in an empty state ( $\epsilon=0$ ), except one, randomly selected, site which is in a prey-predator state ( $\epsilon=3$ ). Numerical results are shown in Fig. 5 and Fig. 6. In these simulations the system is in the active phase ( $r>r_c$ ) and relatively far from the critical point. Although the survival probability  $P(t)$  saturates for large  $t$  its asymptotic values are very low (Fig. 5). For example, to confirm with this method that for  $r=0.12$  the system is in the active phase tens of millions of runs must be made. The reason for that is that for most of the runs predators die out very quickly. Only very few runs generate sufficiently many preys which can support the population of predators. If the population of predators

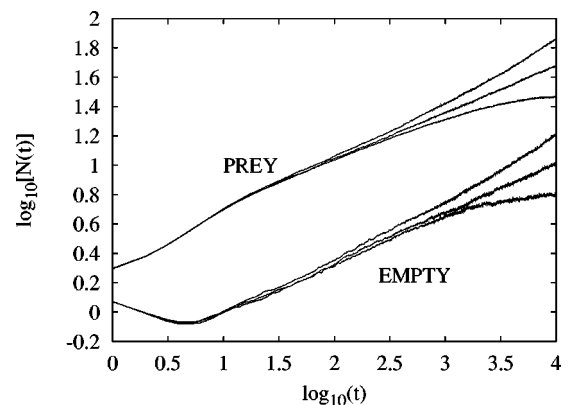


FIG. 7. The average number of predators  $N(t)$  as a function of time  $t$  for the model on the one-dimensional lattice with the absorbing state PREY and EMPTY. For each absorbing state simulations were made for (from top)  $r=0.493, 0.491,$  and  $0.489$ . The slope of the critical (central) lines is very close to the  $(1+1)$  directed percolation value  $\eta=0.314$  [20].

survives for a certain time, then, most likely, it will survive for an infinitely long time spreading through the system (see Fig. 6). We expect that such a behavior exists for any  $r > r_c$ . However, for  $r$  close to  $r_c$  the long-time survival events are extremely unlikely, which renders this method inefficient.

Our model in this case resembles a metastable system [22]. Such a system can exit the metastable state only through formation of a sufficiently large seed of the stable phase. In our case the EMPTY absorbing state corresponds for  $r > r_c$  to a metastable state. Only when a sufficiently large island of the active state occurs, the system is irreversibly driven toward the active (stable) state.

However, this metastable effect is much weaker in one-dimensional systems. In this case the critical point is at  $r = r_c = 0.491(2)$  [17]. Dynamical Monte Carlo simulations for both PREY and EMPTY absorbing states are relatively efficient (see Fig. 7). In the case of the EMPTY absorbing state only a mild decrease of  $N(t)$  is seen for small  $t$ .

We do not fully understand why metastability is much weaker in the  $d=1$  case. Most likely it is due to a much easier formation of a sufficiently large seed of a stable phase. Outward spreading of such an active seed resembles a biased random walk and thus grows steadily in time. In the  $d=2$

case, the formation and growth of the active seed is more complicated.

#### IV. CONCLUSIONS

In the present paper we studied oscillatory and dynamical properties of the two-dimensional prey-predator systems. Our results show that even for the increased range of interactions an amplitude of oscillations vanishes in the thermodynamic limit. It might suggest, as found by Grinstein *et al.* in a related context [12], that oscillations might persist but only for  $d > 2$ .

In addition to that, we examined the validity and efficiency of the dynamical Monte Carlo method for a model with two asymmetric absorbing states. It turns out that for an unstable absorbing state in the two-dimensional system this method is very inefficient. On the other hand, both absorbing states can be used within this method in a one-dimensional model.

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